Lecture 5: Coding of Analog Sources – Sampling and Quantization

Images and sounds are not originally digital! They are continuous signals in space/time as well as amplitude.

Typical model of an analog source: A stationary Gaussian process.

The process can be memoryless (white) or have memory (coloured). See the power spectral density.

**Basic Probability Theory (Sayood App. A)**

- Probability distribution function (pdf) of a stochastic variable $X$:
  \[ P(a \leq X \leq b) = \int_a^b f_X(x) \, dx \]

- Mean / average / expected value of a stochastic variable $X$:
  \[ E[X] = \mu_X = \int_{-\infty}^{\infty} x f_X(x) \, dx \]

- Variance of a stochastic variable $X$:
  \[ \text{Var}[X] = \sigma_X^2 = (x - \mu_X)^2 f_X(x) \, dx \]

**Going to the frequency domain (Sayood 11)**

- The Fourier transform and its inverse:
  \[ X(f) = F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \]
  \[ x(t) = F^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df \]

- Power spectral density (psd):
  \[ \Phi_{xx}(f) = |X(f)|^2 = F[r(\tau)] \]

- Convolution:
  \[ F\{x(t) * y(t)\} = F\{x(t)\} \cdot F\{y(t)\} = X(f) \cdot Y(f) \]
Back to Coding of Analog Sources

Distortion-free coding $\Rightarrow \infty$ bits!

Thus: The rate $R_C$ must be related to the acceptable distortion $D$.

Rate: The number of bits per second.

Distortion: The mean square error $D = \text{E}[(x(t) - \hat{x}(t))^2]$.

How are $R_C$ and $D$ related?

Rate/Distortion

According to Shannon, the following lower limit holds for a white Gaussian process with bandwidth $W$:

$$R_C \geq W \cdot \frac{1}{2} \log \frac{\sigma_X^2}{D} \text{ [bits/s]}$$

Example: Speech signal with $W = 4$ kHz and SNR = 40 dB.

$$10 \cdot \log \frac{\sigma_X^2}{D} = 40 \quad \Rightarrow \quad \frac{\sigma_X^2}{D} = 10^4$$

$$R_C = 4000 \cdot \frac{1}{2} \log 10^4 = 53 \text{ [kbit/s]}$$

Distortion

Definition of distortion: $D = \text{E}[(x(t) - \hat{x}(t))^2]$.

Analog signal: $D = \frac{1}{T} \int (x(t) - \hat{x}(t))^2 dt$

Sampled signal: $D = \frac{1}{N} \sum (x[n] - \hat{x}[n])^2$

Signal-to-Noise ratio (SNR) = $\frac{\sigma_X^2}{D}$

Usually measured in dB: $\text{SNR} = 10 \log \frac{\sigma_X^2}{D} \text{ [dB]}$.

Pulse-Code Modulation (PCM)

Basic idea:
1. Sample the signal $\rightarrow$ time discrete signal.
2. Quantize the samples $\rightarrow$ digital signal
3. Variable length coding $\rightarrow$ more efficient representation

The distortion $D = D_{\text{sampling}} + D_{\text{quantization}}$
**Sampling**

In the frequency domain:

\[ x_{\text{samp}}(t) = x(t) \cdot s(t) = x(t) \cdot \sum_{n} \delta(t - \frac{n}{f_s}) \]

In the frequency domain: \( X_{\text{samp}}(f) = X(f) \ast S(f) \)

**Sampling, cont**

If \( W < f_s/2 \), \( x(t) \) can be reconstructed without distortion.

Apply a low-pass filter removing frequencies higher than \( f_s/2 \)!

If the signal has power outside \( f_s/2 \) the distortion is:

\[ D_{\text{samp}} = \mathbb{E}[(\tilde{x} - x)^2] = 2 \int_{f_s/2}^{\infty} \Phi(f) df \]

Without the lowpass-filtering we get aliasing which doubles the distortion!

**Quantization**

From continuous to discrete alphabet.

Example: 3 bits give 8 levels, 4 bits give 16 levels.

Designing a quantizer: Place the reconstruction levels, the interval limits, and the saturation level.

**Quantization Example**

Original signal

Quantized signal and the error

Coarse quantization
Quantization Distortion

For each bit, the amplitude error is halved, and thus the distortion is decreased with a factor 4, that is: \( D \propto 2^{-2R} \).

\( D \) is also proportional to the signal energy: \( D \propto \sigma_X^2 \).

Thus:
\[ D = c \cdot \sigma_X^2 \cdot 2^{-2R} \]

for some constant \( c \).

Consequently, the rate is:
\[ R = \frac{1}{2} \log \left( \frac{c \cdot \sigma_X^2}{D} \right) \]

Quantizing a Band-limited White Process

Sample at \( f_s = 2W \) samples/s to avoid sampling distortion. Thus, we get \( 2WR \) bits/s.

\[ 2R = \log \left( \frac{c \cdot \sigma_X^2}{D} \right) \Rightarrow 2WR = W \log \left( \frac{c \cdot \sigma_X^2}{D} \right) \]

\[ \Rightarrow R_C = W \log \left( \frac{c \cdot \sigma_X^2}{D} \right) \geq W \log \frac{\sigma_X^2}{D} \]

\[ \Rightarrow c \geq 1 \]

\[ \Rightarrow c_2 \geq 0 \]

Thus \( SNR = 6R \) is the best that can be achieved when quantizing a white Gaussian process.

Quantization Distortion, cont

Rewrite to get the SNR in dB:
\[ \frac{\sigma_X^2}{D} = \frac{1}{c^2} 2^{-2R} \Rightarrow \]
\[ SNR = 10 \cdot \log \frac{\sigma_X^2}{D} = 10 \cdot \log \frac{1}{c^2} 2^{-2R} = 10(2R \log 2 - \log c) \]
\[ = 6R - c_2 \ [dB] \]

\( c \) (and thus \( c_2 = 10 \log c \)) depends on the distribution and the type of quantization. For uniform quantization \( c_2 \) is approximately 7dB (see slide 21).

Example: Digital telephony using 8 bits/sample: \( SNR = 41 \) dB

Practical Quantizers

A practical quantizer is represented by a decisions levels \( d_i \) and reconstruction levels \( r_i \).

The error: \( x - r_i \).

Distortion contribution from the ith interval:
\[ \int_{d_i}^{d_i+1} (x - r_i)^2 f_X(x) \, dx. \]

Total distortion: \( D_{quant} = \sum_{i=1}^{N} \int_{d_i}^{d_i+1} (x - r_i)^2 f_X(x) \, dx. \)
Optimal Quantization ("pdf-Optimized")

\[ \frac{\delta D}{\delta d_i} = 0 \Rightarrow d_i = \frac{r_{i-1} + r_i}{2} \]  
(Check yourself!)

\[ \frac{\delta D}{\delta r_i} = 0 \Rightarrow r_i = \frac{\int_{d_i}^{d_{i+1}} xf_X(x)dx}{\int_{d_i}^{d_{i+1}} f_X(x)dx} \]  
(Center of gravity)

- Numerical solutions by Joel Max in 1960, “Max quantization”

- The table collection give optimal quantizers for 2, 4, 8, ... levels (1, 2, 3, ... bits) and the associated distortion for signals with variance=1 and different distributions (Gauss, Laplace, Rayleigh).

- Note that the step sizes are non-uniform!

The total distortion becomes \( D_{\text{quant}} = \sum p_i \Delta_x^2 = \frac{1}{12} \int \Delta^2(x)f_X(x)dx \), where \( \Delta(x) \) gives the interval length as a function of \( x \).

The number of intervals is then \( N = \int \frac{1}{\Delta(x)} dx \).

If we represent that with a fixed length code we get (the usual) \( R = \log N \) bits.

Fine Quantization

If the number of levels is very large (\( R >> 1 \)) certain approximations can be introduced to give closed-form solutions.

We approximate \( f_X(x) = f_X(d_i) \) (constant within the interval).

Thus, the probability that the signal falls in interval \( i \) is \( p_i = \Delta_i f_X(d_i) \) and the reconstruction level should be

\[ r_i = \frac{d_i + d_{i+1}}{2} \]

The distortion contribution from interval \( i \) becomes

\[ \int_{d_i}^{d_{i+1}} (x - r_i)^2 f_X(x)dx = \int_{-\Delta/2}^{\Delta/2} z^2 f_X(d_i)dz = p_i \int_{-\Delta/2}^{\Delta/2} z^2 dz = p_i \frac{\Delta^2}{12} \]

Case 1: Fine Uniform Quantization

Choose \( a_{\text{max}} \) so that \( P(|X| > a_{\text{max}}) \) is small!

\[ D = \sum p_i \Delta_x^2 = \frac{\Delta^2}{12} \]
Fine Uniform Quantization, cont

From last slide: \[ D = \frac{\Delta^2}{12} \]

Number of levels: \[ N = 2^R = \frac{2a_{\text{max}}}{\Delta} \]

\[ \Rightarrow \Delta = \frac{2a_{\text{max}}}{2^R} \quad \text{and} \quad D = \frac{a_{\text{max}}^2}{3} \cdot 2^{-2R} \]

Typically choose \( a_{\text{max}} = 4\sigma \) \[ \Rightarrow \frac{D}{\sigma^2} = \frac{16}{3} \cdot 2^{-2R} \]

SNR = \( 6R - 7.3 \) [dB]

Case 2: Fine Max-quantization

Also called pdf-optimized quantization or source adapted quantization.

Minimizing \( D \) over all \( D(x) \) and keeping \( N \) constant yields:

\[ \Delta(x) = c \cdot f_x^{-1/3}(x) \]

For Gaussian distribution this gives \[ \frac{D}{\sigma^2} = \frac{\pi\sqrt{3}}{2} \cdot 2^{-2R} \]

\[ \text{SNR} = 6R - 4.34 \, [\text{dB}] \]

Case 3: Quantization + Entropy Coding

Use a variable length code (typically Huffman coding or arithmetic coding) for the intervals.

\[ x[n] \rightarrow \text{Q} \rightarrow \text{VLC} \rightarrow \]

It can be shown that uniform quantization is always best regardless of the probability distribution of \( x \! \)!

For Gaussian distributions: \[ \frac{D}{\sigma^2} = \frac{\pi e}{6} \cdot 2^{-2R} \]

\[ \text{SNR} = 6R - 1.53 \, [\text{dB}] \]

Summary

1. Analog sources are modelled as stochastic processes, for example a Gaussian process with PSD (power spectral density) \( \Phi_{xx}(f) \).
2. The data rate \( r \) must be related to the allowed distortion \( D \).
3. For a white band-limited Gaussian process \( x \sim \mathcal{W} \! \mathcal{N}(0, \sigma^2/2) \) [bits/s).
4. PCM: Sampling + Quantization, \( D = D_{\text{samp}} + D_{\text{quant}} \).
5. \[ D_{\text{samp}} = \frac{\pi e}{6} \cdot 2^{-2R} \]
6. \[ D_{\text{quant}} = \sum_{i=1}^{2^R-1} (i - 2^{R-1})^2 \cdot 2^{R-1} \]
7. Numerical optimization of \( \epsilon_i \) gives Max-quantization (source adapted quantization, pdf-optimized quantization).
8. Fine quantization:  
\[ D = \frac{1}{2} \int_{-\Delta}^{\Delta} g(x) dx, \quad M = \int_{-\Delta}^{\Delta} g(x) dx \]

<table>
<thead>
<tr>
<th>Quantization Type</th>
<th>Expression</th>
<th>SNR</th>
<th>dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fine uniform quantization:</td>
<td>[ D = \frac{1}{2} ] [ \sigma_x^2 ] [ 2^{-m} ]</td>
<td>SNR = 6(R) - 7.3</td>
<td></td>
</tr>
<tr>
<td>Fine Max-quantization:</td>
<td>[ D = \frac{1}{2} ] [ \sigma_x^2 ] [ 2^{-m} ]</td>
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<td></td>
</tr>
<tr>
<td>Fine entropy coded q.:</td>
<td>[ D = \frac{1}{2} ] [ \pi \sigma_x^2 ] [ 2^{-m} ]</td>
<td>SNR = 6(R) - 1.53</td>
<td></td>
</tr>
<tr>
<td>Shannon limit:</td>
<td>[ D = \sigma_x^2 ] [ 2^{-m} ]</td>
<td>SNR = 6(R)</td>
<td></td>
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