Tables and Formulas for Image Coding and Data Compression

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Information Theoretic Concepts

Alphabets and Probabilities

$$X \text{ has alphabet } \{a_1, \dots, a_N\} \ , \quad Y \text{ has alphabet } \{b_1, \dots, b_M\}$$

$$P_X(a_i) = Pr[X = a_i] \ , \ \ P_Y(b_j) = Pr[Y = b_j] \ , \ \ P_{XY}(a_i, b_j) = Pr[X = a_i, Y = b_j]$$

$$P_{X|Y}(a_i|b_j) = Pr[X = a_i|Y = b_j] = \frac{P_{XY}(a_i, b_j)}{P_Y(b_j)}$$

$$P_{Y|X}(b_j|a_i) = Pr[Y = b_j|X = a_i] = \frac{P_{XY}(a_i, b_j)}{P_X(a_i)}$$

Entropy

$$H(X) = -\sum_{i=1}^{N} P_X(a_i) \cdot \log P_X(a_i) \qquad H(Y) = -\sum_{j=1}^{M} P_Y(b_j) \cdot \log P_Y(b_j)$$

$$H(X, Y) = -\sum_{i=1}^{N} \sum_{j=1}^{M} P_{XY}(a_i, b_j) \cdot \log P_{XY}(a_i, b_j)$$

$$H(X|Y) = -\sum_{i=1}^{N} \sum_{j=1}^{M} P_{XY}(a_i, b_j) \cdot \log P_{X|Y}(a_i|b_j)$$

$$H(Y|X) = -\sum_{i=1}^{N} \sum_{j=1}^{M} P_{XY}(a_i, b_j) \cdot \log P_{Y|X}(b_j|a_i)$$

The Chain Rule

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

$$H(X_1 X_2 \dots X_n) = H(X_1) + H(X_2|X_1) + \dots + H(X_n|X_1 \dots X_{n-1})$$

Mutual Information

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

Rate Distortion Bounds

Memoryless Time-discrete Gaussian Process

$$R(D) = \frac{1}{2}\ \log \frac{\sigma^2}{D}$$

Time-discrete Gaussian Process

$$\begin{cases} D = \int_0^1 \min[\Phi[\theta], \gamma] d\theta \\ R = \int_0^1 \max[0, \frac{1}{2}\ \log \frac{\Phi[\theta]}{\gamma}] d\theta \end{cases}$$

Time-continuous Gaussian Process

$$\begin{cases} D = \int_{-\infty}^{\infty} \min[\Phi(f), \gamma] df \\ R_C = \int_{-\infty}^{\infty} \max[0, \frac{1}{2}\ \log \frac{\Phi(f)}{\gamma}] df \end{cases}$$

Quantization

Lloyd-Max Quantization

Optimal choice of decision regions:

$$b_i = \frac{y_{i+1} + y_i}{2}\ ;\quad i = 1, \dots, M-1$$

Optimal choice of reconstruction levels:

$$y_i = \frac{\int_{b_{i-1}}^{b_i} x f(x) dx}{\int_{b_{i-1}}^{b_i} f(x) dx}\ ;\quad i = 1, \dots, M$$

Data rate:

$$R = \log_2 M \quad \text{(No entropy coding is done)}$$

Lloyd-Max quantizers for some distributions

	Gaussian $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m_x)^2}{2\sigma^2}}$ Values for $m_x = 0, \sigma = 1$			Laplacian $f(x) = \frac{1}{\sqrt{2}\sigma} e^{-\frac{\sqrt{2}\lvert x-m_x\rvert}{\sigma}}$ Values for $m_x = 0, \sigma = 1$		
R	b_i	y_i	D	b_i	y_i	D
1	$-\infty$	-0.7979	0.3634	$-\infty$	-0.7071	0.5
	0.0000	0.7979		0.0000	0.7071	
	∞			∞		
2	$-\infty$	-1.5104	0.1175	$-\infty$	-1.8340	0.176
	-0.9816	-0.4528		-1.1269	-0.4198	
	0.0000	0.4528		0.0000	0.4198	
	0.9816	1.5104		1.1269	1.8340	
	∞			∞		
3	$-\infty$	-2.1519	0.03454	$-\infty$	-3.0867	0.0545
	-1.7479	-1.3439		-2.3796	-1.6725	
	-1.0500	-0.7560		-1.2527	-0.8330	
	-0.5005	-0.2451		-0.5332	-0.2334	
	0.0000	0.2451		0.0000	0.2334	
	0.5005	0.7560		0.5332	0.8330	
	1.0500	1.3439		1.2527	1.6725	
	1.7479	2.1519		2.3796	3.0867	
	∞			∞		
4	$-\infty$	-2.7326	0.009497	$-\infty$	-4.4311	0.0154
	-2.4008	-2.0690		-3.7240	-3.0169	
	-1.8435	-1.6180		-2.5971	-2.1773	
	-1.4371	-1.2562		-1.8776	-1.5778	
	-1.0993	-0.9423		-1.3444	-1.1110	
	-0.7995	-0.6568		-0.9198	-0.7287	
	-0.5224	-0.3880		-0.5667	-0.4048	
	-0.2582	-0.1284		-0.2664	-0.1240	
	0.0000	0.1284		0.0000	0.1240	
	0.2582	0.3880		0.2664	0.4048	
	0.5224	0.6568		0.5667	0.7287	
	0.7995	0.9423		0.9198	1.1110	
	1.0993	1.2562		1.3444	1.5778	
	1.4371	1.6180		1.8776	2.1773	
	1.8435	2.0690		2.5971	3.0169	
	2.4008	2.7326		3.7240	4.4311	
	∞			∞		

Multiply reconstruction levels y_i and decision borders b_i by the standard deviation σ and add the mean m_x to get the actual values. The distortion values D should be multiplied by the variance σ^2.

Fine Quantization

$$\begin{cases} D \approx \frac{1}{12}\int_{-\infty}^{\infty}\Delta^2(x)f(x)dx \\ N \approx \int_{-\infty}^{\infty}\frac{1}{\Delta(x)}dx \end{cases}$$

Uniform quantization

$$\Delta(x) = \begin{cases} \Delta & a_{min} \leq x \leq a_{max} \\ \infty & \text{otherwise} \end{cases}$$

$$D \approx \frac{\Delta^2}{12}$$

Logarithmic quantization

$$\Delta(x) = \begin{cases} k|x| & a_{min} \leq |x| \leq a_{max} \\ \infty & \text{otherwise} \end{cases}$$

Source adapted quantization (fine Lloyd-Max quantization)

$$\Delta(x) = c \cdot (f(x))^{-\frac{1}{3}}$$

Fine Quantization of Gaussian Distributions

Source adapted quantization (fine Lloyd-Max quantization)

$$D \approx \frac{\pi\sqrt{3}}{2} \cdot \sigma^2 \cdot 2^{-2R}$$

Uniform quantization followed by entropy coding ($R = H(\hat{X})$)

$$D \approx \frac{\pi e}{6} \cdot \sigma^2 \cdot 2^{-2R}$$

Predictive Coding

Optimal (LMS) linear predictor

$$p_n = a_1 \hat{x}_{n-1} + \ldots + a_N \hat{x}_{n-N} \approx a_1 x_{n-1} + \ldots + a_N x_{n-N} = \mathbf{A} \cdot \mathbf{x}$$

$$\mathbf{A} = \mathbf{R}^{-1} \mathbf{P}$$

where

$$\mathbf{A} = (a_1 \ ... \ a_N)^T \ , \quad \mathbf{x} = (x_{n-1} \ ... \ x_{n-N})^T$$

$$\mathbf{P} = E\{x_n \mathbf{x}\} = (E\{x_n x_{n-1}\} \ ... \ E\{x_n x_{n-N}\})^T$$

$$\mathbf{R} = E\{\mathbf{x}\mathbf{x}^T\} = \begin{pmatrix} E\{x_{n-1}x_{n-1}\} & \cdots & E\{x_{n-1}x_{n-N}\} \\ \vdots & \ddots & \vdots \\ E\{x_{n-N}x_{n-1}\} & \cdots & E\{x_{n-N}x_{n-N}\} \end{pmatrix}$$

Prediction error variance for an optimal predictor

$$\sigma_d^2 = \sigma_x^2 - \mathbf{P} \cdot \mathbf{A} = \sigma_x^2 - \mathbf{P}^T \mathbf{A}$$

Transform Coding

$$\theta = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{x} = (x_0 \ ... \ x_{n-1})^T \ , \quad \theta = (\theta_0 \ ... \ \theta_{n-1})^T$$

$$\mathbf{R}_X = E\{\mathbf{x}\mathbf{x}^T\} \ , \quad \mathbf{R}_\theta = E\{\theta\theta^T\} = \mathbf{A}\mathbf{R}_X\mathbf{A}^T$$

The Karhunen-Loève (KL) Transform

The rows of **A**, ie the transform basis functions, are the normalized eigenvectors of the correlation matrix $\mathbf{R}_X$. The transform components will be uncorrelated, ie $\mathbf{R}_\theta$ will be a *diagonal* matrix.

The Hadamard Transform

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{H}_{2n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{H}_n & \mathbf{H}_n \\ \mathbf{H}_n & -\mathbf{H}_n \end{pmatrix}$$

Usually the basis vectors are sorted in frequency order.

The Cosine Transform

$$\mathbf{C}_n = \sqrt{\frac{2}{n}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \cos\frac{\pi}{2n} & \cos\frac{3\pi}{2n} & \cdots & \cos\frac{(2n-1)\pi}{2n} \\ \cos\frac{2\pi}{2n} & \cos\frac{6\pi}{2n} & \cdots & \cos\frac{2(2n-1)\pi}{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \cos\frac{(n-1)\pi}{2n} & \cos\frac{(n-1)3\pi}{2n} & \cdots & \cos\frac{(n-1)(2n-1)\pi}{2n} \end{pmatrix}$$

or equivalently

$$\theta_i = \sqrt{\frac{2}{n}} S_i \sum_{j=0}^{n-1} x_j \cos\frac{(2j+1)i\pi}{2n} \quad ; \quad i = 0, \ldots, n-1$$

$$S_i = \begin{cases} \frac{1}{\sqrt{2}} & ; \quad i = 0 \\ 1 & ; \quad \text{otherwise} \end{cases}$$

Quantization of Transform Components

Optimal bit allocation when using *fine quantization*. Assume that the variance of transform component i is σ_i^2.

$$R_i = R + \frac{1}{2}\log\frac{\sigma_i^2}{\sigma_G^2}$$

where

$$R = \frac{1}{n}\sum_{i=0}^{n-1} R_i$$

and

$$\sigma_G^2 = \sqrt[n]{\prod_{i=0}^{n-1} \sigma_i^2}$$

gives

$$D \approx c \cdot \sigma_G^2 \cdot 2^{-2R}$$

For the KL transform

$$\sigma_G^2 = (\det \mathbf{R}_X)^{\frac{1}{n}} = (\det \mathbf{R}_\theta)^{\frac{1}{n}}$$