Network Information Theory
(Channel Capacity)
Part 2

Muhammad Fahim Ul Haque
Computer Engineering Division
EE Department
Linköping University
Content of Presentation

• Multiple Access Channel (Continued)
  Gausian Multi Access Channel
  Gausian Channel Capacity for FDMA,TDMA &CDMA)
  m-user Multi Access Channel

• Broadcast Channel
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• Relay Channel
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Gaussian Multiple Acces Channel

• Generalize case for capacity region for multiple acces channel is given as:

\[ R_1 \leq I(X_1; Y|X_2), \quad (1) \]
\[ R_2 \leq I(X_2; Y|X_1), \quad (2) \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y) \quad (3) \]

• Capacity region for gausian channel drived from above equation
Gaussian Multiple Access Channel

From equation 1

\[ I(X_1; Y|X_2) = h(Y|X_2) - h(Y|X_1X_2) \] \hspace{1cm} (4)

\[ = h(X_1 + X_2 + Z|X_2) - h(X_1 + X_2 + Z|X_1X_2) \] \hspace{1cm} (5)

\[ = h(X_1 + Z|X_2) + h(Z|X_1X_2) \] \hspace{1cm} (6)

\[ = h(X_1 + Z|X_2) - h(Z) \] \hspace{1cm} (7)

\[ = h(X_1 + Z) - h(Z) \] \hspace{1cm} (8)

\[ = \frac{1}{2} \log(2\pi e) (P_1 + N) - \frac{1}{2} \log(2\pi e) N \] \hspace{1cm} (9)

\[ I(X_1; Y|X_2) = \frac{1}{2} \log \left( 1 + \frac{P_1}{N} \right) \] \hspace{1cm} (10)
Gaussian Multiple Access Channel

\[ I(X_2; Y|X_1) = h(Y|X_1) - h(Y|X_1X_2) \]  \hspace{1cm} (11)
\[ = \frac{1}{2} \log(2\pi e) (P_2 + N) - \frac{1}{2} \log(2\pi e) N \]  \hspace{1cm} (12)
\[ I(X_2; Y|X_1) = \frac{1}{2} \log \left(1 + \frac{P_2}{N}\right) \]  \hspace{1cm} (13)

\[ I(X_1, X_2; Y) = h(Y) - h(Y|X_1X_2) \]  \hspace{1cm} (14)
\[ = h(X_1 + X_2 + Z) - h(Z) \]  \hspace{1cm} (15)
\[ = \frac{1}{2} \log(2\pi e) (P_2 + N) - \frac{1}{2} \log(2\pi e) N \]  \hspace{1cm} (16)
\[ I(X_1, X_2; Y) = \frac{1}{2} \log \left(1 + \frac{P_1 + P_2}{N}\right) \]  \hspace{1cm} (17)
Gaussian Multiple Access Channel

• So

\[ R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{N} \right) \]  \hspace{1cm} (18)

\[ R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_2}{N} \right) \]  \hspace{1cm} (19)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2}{N} \right) \]  \hspace{1cm} (20)
Gaussian Multiple Access Channel

- FDMA
- TDMA
- CDMA
m- user Channel

\[ R(S) \leq I(X(S) ; Y | X(S^c)) \quad \text{for all } S \subseteq \{1, 2, \ldots, m\} \]

\[ \sum_{i \in S} R_i = \text{total rate of information flow from } S \leq C \left( \frac{\sum_{i \in S} P_i}{N} \right). \]
Broad Cast Channel

$$p(y_1, y_2 | x)$$

Encoder

Decoder

Hat W_1

Hat W_2

(W_1, W_2)
Examples Of Broadcast Channel

- TV Broadcast Station
- Class Room Lecture
- TV Broadcast Station (Simultaneous HDTV and Regular Transmission)
- Orthogonal Broadcast Station.
Broadcast Channel

- A broadcast channel with two receivers of rate \( R_1 \) and \( R_2 \) and there is no common info. sent to both receivers

\[
X : (\{1, 2, \ldots, 2^{nR_1}\} \times \{1, 2, \ldots, 2^{nR_2}\}) \to \mathcal{X}^n
\]

\[
g_1 : \mathcal{Y}_1^n \to \{1, 2, \ldots, 2^{nR_1}\}
\]

\[
g_2 : \mathcal{Y}_2^n \to \{1, 2, \ldots, 2^{nR_2}\}
\]

\[
P_e^{(n)} = P(g_1(Y_1^n) \neq W_1 \text{ or } g_2(Y_2^n) \neq W_2),
\]
Broadcast Channel

• A rate \((R_1, R_2)\) is said to be achievable for the broadcast channel if there exists a sequence of \(((2^{nR_1}, 2^{nR_2}), n)\) codes with \(P_e^{(n)} \to 0\).

• The channel where we have common info. Of rate \(R_0\) to be sent to both receiver.

\[
X : (\{1, 2, \ldots, 2^{nR_0}\} \times \{1, 2, \ldots, 2^{nR_1}\} \times \{1, 2, \ldots, 2^{nR_2}\}) \to \chi^n
\]

\[
g_1 : \mathcal{Y}_1^n \to \{1, 2, \ldots, 2^{nR_0}\} \times \{1, 2, \ldots, 2^{nR_1}\}
\]

\[
g_2 : \mathcal{Y}_2^n \to \{1, 2, \ldots, 2^{nR_0}\} \times \{1, 2, \ldots, 2^{nR_2}\}.
\]

\[
P_e^{(n)} = P(g_1(Y_1^n) \neq (W_0, W_1) \text{ or } g_2(Z^n) \neq (W_0, W_2)).
\]
Broadcast Channel

- A rate triple \((R_0, R_1, R_2)\) is said to achievable for broadcast channel with common info. If there exist a sequence of \(((2^{nR_0}, 2^{nR_1}, 2^{nR_2}), n)\) codes with \(P_e^{(n)} \to 0\).

- The capacity region of the broadcast channel is the closure of the set of achievable rates.

- **Theorem 15.6.1:** The capacity region of a broadcast channel depends only on the conditional marginal distributions \(p(y_1 | x)\) and \(p(y_2 | x)\).
Degraded Broadcast Channels

- A broadcast channel is said to be physically degrades if
  \[ p(y_1, y_2|x) = p(y_1|x)p(y_2|y_1) \]

- A broadcast channel is said to be stochastically degrade if its conditional marginal distribution are same as that of physically degraded broadcast channel
  \[ p(y_2|x) = \sum_{y_1} p(y_1|x)p'(y_2|y_1) \]
Capacity of Degraded Broadcast Channel

**Theorem 15.6.2**  The capacity region for sending independent information over the degraded broadcast channel $X \to Y_1 \to Y_2$ is the convex hull of the closure of all $(R_1, R_2)$ satisfying

$$R_2 \leq I(U; Y_2),$$  \hspace{1cm} (15.210)

$$R_1 \leq I(X; Y_1|U)$$  \hspace{1cm} (15.211)

for some joint distribution $p(u)p(x|u)p(y_1, y_2|x)$, where the auxiliary random variable $U$ has cardinality bounded by $|U| \leq \min\{|X|, |Y_1|, |Y_2|\}$. 
Capacity of Degraded Broadcast Channel

• Proof

\[ E_{Yi} = \{(U(i), Y_2) \in A^{(n)}_{\epsilon}\}. \]  

\[ P_e^{(n)}(2) = P(E_{Y1}^c \bigcup \bigcup_{i \neq 1} E_{Yi}) \]  

\[ \leq P(E_{Y1}^c) + \sum_{i \neq 1} P(E_{Yi}) \]  

\[ \leq \epsilon + 2^{nR_2} 2^{-n(I(U;Y_2)-2\epsilon)} \]  

\[ \leq 2\epsilon \]
Capacity of Degraded Broadcast Channel

\[ \tilde{E}_{Y_i} = \{(U(i), Y_1) \in A_{\varepsilon}^{(n)}\}, \]  

(15.217)

\[ \tilde{E}_{Y_{ij}} = \{(U(i), X(i, j), Y_1) \in A_{\varepsilon}^{(n)}\}, \]  

(15.218)

\[ P_e^{(n)}(1) = P \left( \tilde{E}_{Y_1}^c \bigcup \tilde{E}_{Y_{11}}^c \bigcup \bigcup_{i \neq 1} \tilde{E}_{Y_i} \bigcup \bigcup_{j \neq 1} \tilde{E}_{Y_{1j}} \right) \]  

(15.219)

\[ \leq P(\tilde{E}_{Y_1}^c) + P(\tilde{E}_{Y_{11}}^c) + \sum_{i \neq 1} P(\tilde{E}_{Y_i}) + \sum_{j \neq 1} P(\tilde{E}_{Y_{1j}}). \]  

(15.220)

\[ P_e^{(n)}(1) \leq \varepsilon + \varepsilon + 2^{nR_2}2^{-n(I(U; Y_1) - 3\varepsilon)} + 2^{nR_1}2^{-n(I(X; Y_1|U) - 4\varepsilon)} \leq 4\varepsilon \]  

(15.227)
Theorem 15.6.3  If the rate pair \((R_1, R_2)\) is achievable for a broadcast channel with independent information, the rate triple \((R_0, R_1 - R_0, R_2 - R_0)\) with a common rate \(R_0\) is achievable, provided that \(R_0 \leq \min(R_1, R_2)\).

Theorem 15.6.4  If the rate pair \((R_1, R_2)\) is achievable for a degraded broadcast channel, the rate triple \((R_0, R_1, R_2 - R_0)\) is achievable for the channel with common information, provided that \(R_0 < R_2\).
Gaussian Broadcast Degraded Channel

\[ Z_1 \sim \mathcal{N}(0, N_1) \quad Z_2' \sim \mathcal{N}(0, N_2 - N_1) \]

\[ R_1 < C \left( \frac{\alpha P}{N_1} \right) \]

\[ R_2 < C \left( \frac{(1 - \alpha) P}{\alpha P + N_2} \right) \]
Relay Channel
Relay Channel

**Definition** A \((2^n R, n)\) code for a relay channel consists of a set of integers \(\mathcal{W} = \{1, 2, \ldots, 2^n R\}\), an encoding function

\[
X : \{1, 2, \ldots, 2^n R\} \rightarrow \mathcal{X}^n, \tag{15.242}
\]

a set of relay functions \(\{f_i\}_{i=1}^{n}\) such that

\[
x_{1i} = f_i(Y_{11}, Y_{12}, \ldots, Y_{1i-1}), \quad 1 \leq i \leq n, \tag{15.243}
\]

and a decoding function,

\[
g : \mathcal{Y}^n \rightarrow \{1, 2, \ldots, 2^n R\}. \tag{15.244}
\]
Relay Channel

\[ p(w, x, x_1, y, y_1) = p(w) \prod_{i=1} p(x_i|w) p(x_{1i}|y_{11}, y_{12}, \ldots, y_{1i-1}) \]
\[ \times p(y_i, y_{1i}|x_i, x_{1i}). \]

(15.245)

\[ \lambda(w) = \Pr\{g(Y) \neq w | w \text{ sent}\} \]

(15.246)

\[ P_e^{(n)} = \frac{1}{2nR} \sum_{w} \lambda(w). \]

(15.247)
Relay Channel

**Theorem 15.7.1**  For any relay channel \((\mathcal{X} \times \mathcal{X}_1, p(y, y_1|x, x_1), \mathcal{Y} \times \mathcal{Y}_1)\), the capacity \(C\) is bounded above by

\[
C \leq \sup_{p(x,x_1)} \min \{I(X, X_1; Y), I(X; Y, Y_1|X_1)\}. \tag{15.248}
\]

**Definition**  The relay channel \((\mathcal{X} \times \mathcal{X}_1, p(y, y_1|x, x_1), \mathcal{Y} \times \mathcal{Y}_1)\) is said to be **physically degraded** if \(p(y, y_1|x, x_1)\) can be written in the form

\[
p(y, y_1|x, x_1) = p(y_1|x, x_1)p(y|y_1, x_1). \tag{15.249}
\]

**Theorem 15.7.2**  The capacity \(C\) of a physically degraded relay channel is given by

\[
C = \sup_{p(x,x_1)} \min \{I(X, X_1; Y), I(X; Y_1|X_1)\}, \tag{15.250}
\]
References

• Element of Information Theory (Second Edition) by Thomas M. Cover & Joy A. Thomas